Transverse effects on squeezing with atoms

Astrid Lambrecht, Jean-Michel Courty and Serge Reynaud

Laboratoire Kastler Brossel, UPMC, ENS, CNRS
Université Pierre et Marie Curie, case 74, F75252 Paris, France

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Abstract

We evaluate the squeezing of a probe beam with a transverse Gaussian structure interacting with an ensemble of two-level atoms in a cavity. We use the linear input-output formalism where the effect of atoms is described by susceptibility and noise functions, and show that the transverse structure is accounted for by averaging these atomic functions over the intensity profile. The results of the plane-wave and Gaussian-wave theories are compared. We find that, when a large squeezing is predicted, the prediction of the plane-wave model is not reliable outside the Kerr domain. We give an estimate of the squeezing degradation due to the Gaussian transverse structure.

1 Introduction

When an optical cavity containing a nonlinear medium approaches a bistability turning point [1], the fluctuations of the incoming beam are squeezed [2]. Nearly perfect squeezing is expected when the nonlinear medium is a lossless Kerr medium [3, 4]. Squeezing was experimentally demonstrated at the output of a bistable cavity containing an atomic beam [5, 6] or a cloud of cold atoms released from a magneto-optical trap [7]. A lot of theoretical work has been devoted to the case where the nonlinear medium consists in two-level atoms [8, 9, 10, 11, 12, 13, 14, 15, 16].

Most theoretical treatments have simplified the description of the field-matter interaction by modelizing the laser beam as a plane-wave. It is however clear that the description of the bistable behaviour of the cavity has to be modified to account for the transverse Gaussian profile of the laser beam [17, 18, 19, 20]. This is also true for the predictions concerning field fluctuations at the output of the cavity, as shown in references [21, 22, 23]. These papers contain a derivation of squeezing spectra in optical bistability accounting for the transverse Gaussian structure of the laser beam. For particular sets of parameters, the spectra obtained in Gaussian-wave calculations are compared by the authors with the predictions of the plane-wave model. In the good-cavity

limit where the atomic damping occurs on a much shorter time than the intracavity field damping and where the atomic variables can be adiabatically eliminated, Xiao et al [21] conclude that the plane-wave calculations may be trusted in either the weak-field or dispersive limits. In a complete treatment without adiabatic elimination [23], Hope et al relate squeezing degradation due to transverse structure to the ratio of atomic and cavity damping times.

In the present paper, we quantitatively delineate the regions in the parameter space where the predictions of the plane-wave and Gaussian-wave calculations coincide, and therefore estimate the reliability of plane-wave model. We obtain criteria for reliability of plane-wave model which are quite different from those of references [21, 22, 23].

We will use the linear input-output formalism generalized to incorporate the treatment of atomic fluctuations [16]. This formalism is based upon a linearisation of the fluctuations around the working point of the bistable cavity, as the quantum stochastic methods [24], and it provides the same spectra as the latter when the same system is studied with the same simplifying assumptions [25]. In contrast with the quantum stochastic methods, it provides an intuitive understanding of the noise processing occuring in the bistable cavity. The squeezing process through the parametric transformation of vacuum fluctuations going into the intracavity mode is described in terms of atomic susceptibility functions, while the squeezing degradation through the addition of fluctuations by the atoms is characterized by atomic noise functions.

We will show in the following that the effect of Gaussian profile upon noise processing appears quite naturally as a spatial averaging of these susceptibility and noise functions. This will allow us to obtain simple techniques for describing the influence of transverse structure. It is worth stressing that the property of averaging of the susceptibility and noise functions is not postulated a priori, but is rather a consequence of the simple assumptions made in the treatment of Gaussian profile: we assume that the higher-order transverse modes are either far from resonance with the cavity, or of such a high order that they are not efficiently coupled to the fundamental mode. It is thus possible to project any field radiated by the atoms upon this mode, and to obtain the averaging property as a consequence of this projection.

Using this technique, we will first recover the well-known result [17, 18, 19, 20] that the mean fields evaluated with the plane-wave and Gaussian-wave calculations coincide in the low-saturation limit where the population of the atomic excited state remains small. This implies that the bistability curves relating the input and intracavity intensities also coincide provided that these intensities are properly defined. We will then evaluate the atomic susceptibility and noise functions, and deduce that the conditions for coincidence of plane-wave and Gaussian-wave calculations are much more restrictive in this case: the limit of a small population in the excited state is not sufficient for ensuring the coincidence for atomic noise functions. In particular, we will conclude that the noise functions evaluated in the plane-wave model are not reliable when

large squeezing is predicted with parameters lying outside the Kerr domain. We will finally give a quantitative estimate for the degradation of squeezing due to transverse structure.

2 Model and assumptions

We consider a single ended ring cavity containing a nonlinear medium. A probe beam is entering the cavity through the coupling mirror, interacting with the medium and leaving the cavity through the same mirror. To study the influence of this device on the quantum fluctuations of the probe beam, we use a semi-classical input-output formalism, in which the field fluctuations are considered as classical stochastic variables, which are driven by the quantum fluctuations of the incoming fields at the laser frequency ω_L. The linear input-output relations give the quantum fluctuations of the output field as a function of those of the input field, in cases when they are small compared to the mean field-values, an assumption which is particularly well adapted for high photon numbers, i.e. laser beams in a resonator. The method is described in detail in [16].

The intracavity field is defined as Heisenberg operator in the frame rotating with respect to the laser frequency ω_L:

$$e^{-i\omega_L t} E(\mathbf{r},t) + e^{i\omega_L t} E(\mathbf{r},t)^\dagger \tag{1}$$

In order to account for the spatial beam structure, the components $E(\mathbf{r},t)$ and $E(\mathbf{r},t)^\dagger$ are written as products of a Gaussian mode function by a time-dependent mode amplitude ($\mathbf{r}$ stands for the three-dimensional position (x,y,z)):

$$E(\mathbf{r},t) = \sqrt{\frac{\hbar\omega_L}{2\varepsilon_0 c}} A(t)\, u(\mathbf{r}) \tag{2}$$

$$E(\mathbf{r},t)^\dagger = \sqrt{\frac{\hbar\omega_L}{2\varepsilon_0 c}} A(t)^\dagger\, u(\mathbf{r})^* \tag{3}$$

This definition is such that $\left\langle A(t)^\dagger A(t) \right\rangle$ is the number of photons going through a beam section in the unit time.

The normalized Gaussian mode functions are written for a propagation along the x direction:

$$u(\mathbf{r}) = \sqrt{\frac{f(\mathbf{r})}{S(x)}} \exp\left[-i\varphi(\mathbf{r})\right] \tag{4}$$

$f(r)$ is the Gaussian intensity normalized to unity on the beam axis, S the effective beam section and $\varphi(r)$ the phase:

$$f(\mathbf{r}) = \exp\left[\frac{-2\left(y^2+z^2\right)}{w^2(x)}\right] \quad (5)$$

$$S(x) = \frac{\pi w^2(x)}{2} \quad (6)$$

$$\varphi(\mathbf{r}) = -\frac{2\pi x}{\lambda} + \arctan\left(\frac{x}{l_R}\right) - \frac{\pi}{\lambda}\frac{\left(y^2+z^2\right)x}{x^2+l_R^2} \quad (7)$$

$w(x)$ denotes the position-dependent beam size, w_0 the beam waist, l_R the Rayleigh divergence length and λ the laser wavelength:

$$w^2(x) = w_0^2\left(1+\frac{x^2}{l_R^2}\right) \quad (8)$$

$$l_R = \frac{\pi w_0^2}{\lambda} \quad (9)$$

Since we suppose the cavity to be perfectly mode matched to the laser beam, the same transverse structure holds for the input field and for the intracavity field before its interaction with the atoms. This structure is modified for the output field however, through the non linear interaction with the atoms on one hand, and through spontaneous emission of light in all possible directions by the atoms on the other hand. Both effects couple the fundamental Gaussian mode to all the other cavity modes. We will suppose here that the field radiated by the atoms into the other modes has an effect on intracavity fields much smaller than the field radiated into the fundamental Gaussian mode. This may be due either to the fact that the fundamental Gaussian mode is close to a cavity resonance while the other modes at the same frequency are far from any cavity resonance, or that they have such a high order that their coupling with the fundamental mode is negligeable. We will also consider that the detection setup is perfectly matched to the Gaussian mode.

As a consequence, we can write the input and output fields $E^{\text{in}}(\mathbf{r},t)$ and $E^{\text{out}}(\mathbf{r},t)$ in the same manner as the intracavity field $E(\mathbf{r},t)$:

$$E^{\text{in}}(\mathbf{r},t) = \sqrt{\frac{\hbar\omega_L}{2\varepsilon_0 c}}A^{\text{in}}(t)\,u(\mathbf{r}) \quad (10)$$

$$E^{\text{out}}(\mathbf{r},t) = \sqrt{\frac{\hbar\omega_L}{2\varepsilon_0 c}}A^{\text{out}}(t)\,u(\mathbf{r}) \quad (11)$$

Clearly the reflection-transmission equations for the fields on the coupling mirror will be the same for the field amplitudes as for the plane-wave fields in [16]. In contrast, the effect of the atoms on these field amplitudes will be modified, due

to the non-linearity of the interaction and to the Gaussian variation of laser intensity. In order to evaluate the modification of the mode amplitudes due to the atomic medium, we will disregard the fields radiated by the atoms into the higher-order modes and obtain these modifications by projecting the radiated field onto the fundamental mode. This will be the key simplifying assumption in the forthcoming calculations concerning the mean fields as well as the field fluctuations.

We will also use two simplifying hypothesis. First, we will consider that the transverse size w of the probe beam is much smaller than the size of the atomic medium, so that the dependence of the atomic density versus the transverse variables y and z may be disregarded. This density $\rho(\mathbf{r})$ will therefore be replaced by the on-axis atomic density ρ_{m}:

$$\rho_{\mathrm{m}}(x) = \rho(x, y = 0, z = 0) \tag{12}$$

Note that the longitudinal dependence of the atomic density will have to be accounted for, since we consider that the length of the interaction zone is limited by the size of the atomic medium. Second, we will assume that the Rayleigh divergence length l_R is much larger than this size, so that the x-dependence of the laser beam size w can be ignored. However, we will not suppose that the atomic medium is centered on the cavity waist.

3 Description of mean fields in a Gaussian-wave theory

The atomic medium is considered to be a homogeneously broadened system of two-level atoms with a resonance frequency ω_0, which is placed inside an optical cavity and driven by a coherent mean field of frequency ω_L close to a cavity mode ω_C. Atom-laser detuning δ and cavity-laser detuning ϕ are respectively normalised by the atomic dipole decay rate γ and the cavity field decay rate κ:

$$\delta = \frac{\omega_0 - \omega_L}{\gamma} \tag{13}$$

$$\phi = \frac{\omega_C - \omega_L}{\kappa} \tag{14}$$

It follows from the discussions of the previous section that the evolution of the intracavity Gaussian mode amplitude during one cavity round trip time is described by the same differential equation as in the plane-wave model:

$$\tau_C \partial_t A(t) = -\kappa\tau_C \left(1 + i\phi\right) A(t) + \sqrt{2\kappa\tau_C} A^{\mathrm{in}}(t) + dA(t) \tag{15}$$

However, the modification dA of the intracavity Gaussian mode amplitude due to one passage through the atomic medium has not the same expression as

in the plane-wave model, as a consequence of the nonlinearity of atoms-field interaction. In the present section, we evaluate its mean value and then deduce the bistability curve.

The effect of an optically thin atomic layer of length dx may be described by a local modification $dE(\mathbf{r})$ of the field:

$$E(\mathbf{r}) \rightarrow E(\mathbf{r}) + dE(\mathbf{r}) \tag{16}$$

with [26]:

$$dE(\mathbf{r}) = -\frac{3\lambda^2}{4\pi}\rho(\mathbf{r})\,\alpha(\mathbf{r})\,E(\mathbf{r})\,dx \tag{17}$$

$\alpha(\mathbf{r})$ is the atomic polarizability derived from the solution of the optical Bloch equations [16] and measured as a dimensionless number:

$$\alpha(\mathbf{r}) = \frac{\alpha_l}{1+2s(\mathbf{r})} \tag{18}$$

α_l is the dimensionless linear polarizability of the atomic dipole:

$$\alpha_l = \frac{1}{1+i\delta} \tag{19}$$

$s(\mathbf{r})$ is the dimensionless saturation parameter proportional to the local laser intensity:

$$s(\mathbf{r}) = \frac{|\beta(\mathbf{r})|^2}{1+\delta^2} \tag{20}$$

$\beta(\mathbf{r})$ is the dimensionless field parameter, that is precisely the Rabi frequency normalized to the decay rate of the dipole:

$$\beta(\mathbf{r}) = \frac{d_0 E(\mathbf{r})}{\hbar\gamma} = \sqrt{\frac{3\lambda^2}{4\pi}}\frac{A}{\sqrt{\gamma}}u(\mathbf{r}) \tag{21}$$

where d_0 is the matrix element of the atomic dipole. These relations may also be written in terms of the saturation parameter s_m and of the field parameter β_m evaluated on the beam axis:

$$s(\mathbf{r}) = s_\mathrm{m} f(\mathbf{r}) \tag{22}$$

$$s_\mathrm{m} = \frac{|\beta_\mathrm{m}|^2}{1+\delta^2} \tag{23}$$

$$\beta_\mathrm{m} = \sqrt{\frac{3\lambda^2}{4\pi S}}\frac{A}{\sqrt{\gamma}} \tag{24}$$

Note that we may forget the x-dependence of the on-axis saturation parameter s_m as a consequence of the assumption of a large Rayleigh divergence length.

The more general case of an arbitrary Rayleigh divergence length might as well be studied [27]. Using the transverse monomode approximation discussed above, we project dE onto the Gaussian mode, and write the following equation for the modification dA of the mode amplitude, summing up over the layers inside the atomic medium:

$$\frac{dA}{A} = -\frac{3\lambda^2}{4\pi S}\alpha_l n_s \tag{25}$$

n_s is the effective number of atoms in the probe beam, taking into account the effect of the saturation:

$$n_\mathrm{s} = \int \frac{f(\mathbf{r})}{1 + 2s_\mathrm{m} f(\mathbf{r})} \rho(\mathbf{r})\, d\mathbf{r} \tag{26}$$

In order to evaluate n_s, we expand it in powers of the saturation parameter:

$$n_\mathrm{s} = \sum_{k=0}^{\infty} (-2s_\mathrm{m})^k n^{(1+k)} \tag{27}$$

$$n^{(1+k)} = \int f(\mathbf{r})^{1+k} \rho(\mathbf{r})\, d\mathbf{r} \tag{28}$$

$n^{(1)}$, that we will denote simply n, is the effective number of atoms in the probe beam, defined in the unsaturated regime:

$$n = \int f(\mathbf{r}) \rho(\mathbf{r})\, d\mathbf{r} \tag{29}$$

The next term $n^{(2)}$ corresponds to the lowest-order saturation effect, that is precisely the Kerr nonlinearity of the atomic medium. The other terms represent higher-order saturation effects. Due to the Gaussian shape of the function $f(\mathbf{r})$, the function $f(\mathbf{r})^{1+k}$ has the same expression as $f(\mathbf{r})$ with a modified value of the beam size parameter, precisely with w^2 replaced by $w^2/(1+k)$; $n^{(1+k)}$ has therefore the same expression as n with w^2 replaced by $w^2/(1+k)$.

We now use the assumption that the transverse size w of the probe beam is much smaller than the size of the atomic medium. The unsaturated effective number n has thus a simple expression in terms of the on-axis atomic density:

$$n = \frac{\pi w^2}{2} \int \rho_\mathrm{m}(x)\, dx \tag{30}$$

The higher-order integrals $n^{(1+k)}$ may therefore be written as:

$$n^{(1+k)} = \frac{n}{1+k} \tag{31}$$

The saturated effective number n_{s} can then be related to the unsaturated effective number n through an analytical expression:

$$n_{\mathrm{s}} = nF\left(s_{\mathrm{m}}\right) \tag{32}$$

$$F\left(s_{\mathrm{m}}\right) = 1 - \frac{2s_{\mathrm{m}}}{2} + \frac{\left(2s_{\mathrm{m}}\right)^2}{3} - \frac{\left(2s_{\mathrm{m}}\right)^3}{4} + .. = \frac{1}{2s_{\mathrm{m}}}\log\left(1 + 2s_{\mathrm{m}}\right) \tag{33}$$

This may be compared with the corresponding equation in the plane-wave model:

$$n_{\mathrm{s}} = nF_{\mathrm{pw}}\left(s\right) \tag{34}$$

$$F_{\mathrm{pw}}\left(s\right) = 1 - 2s + \left(2s\right)^2 - \left(2s\right)^3 + .. = \frac{1}{1+2s} \tag{35}$$

The two expressions nearly coincide in the low saturation regime, where only the linear index and the Kerr term are appreciable, provided that an effective value s_{e} of the saturation parameter is chosen in the plane-wave model such that:

$$s_{\mathrm{m}} = 2s_{\mathrm{e}} \tag{36}$$

In more physical terms, this condition means that, in the Kerr limit, the plane-wave and Gaussian results have to be compared for equal mean intensities, i.e. equal intensities after averaging over the Gaussian profile. As soon as the saturation parameter is appreciable in contrast ($s \approx 1$ or $s > 1$), the two expressions differ. More strikingly, the expression (33) which accounts for transverse mode profile does not tend towards the plane-wave expression (35) when the beam size parameter grows. In this regime also, valid conclusions about the effect of gaussian beam structure may be reached only after a criterium has been chosen to compare plane-wave and Gaussian-beam spectra. In the following, we will always compare spectra with equation (36) obeyed.

The modification dA of the intracavity Gaussian mode amplitude due to one passage in the atomic medium may now be written:

$$\frac{dA}{A} = -\frac{3\lambda^2}{4\pi S}\frac{nF\left(s_{\mathrm{m}}\right)}{1+i\delta} \tag{37}$$

We easily obtain the steady state solution of the differential equation (15) for the intracavity amplitude A:

$$\sqrt{2\kappa\tau_C}A^{\mathrm{in}} = \kappa\tau_C\left(1+i\phi\right)A - dA \tag{38}$$

which may be written as:

$$\frac{2}{\sqrt{T}}\beta^{\mathrm{in}} = \beta_{\mathrm{m}}\left[\left(1+i\phi\right) + 2C\frac{F\left(s_{\mathrm{m}}\right)}{1+i\delta}\right] \tag{39}$$

β_{m} denotes the dimensionless parameter for the on-axis intracavity field (cf. equation(24)) and β^{in} a dimensionless parameter for the input field defined in an analogous manner:

$$\beta^{\mathrm{in}} = \sqrt{\frac{3\lambda^2}{4\pi S}} \frac{A^{\mathrm{in}}}{\sqrt{\gamma}} \tag{40}$$

C is the cooperativity parameter, which is proportional to the effective number n of atoms present in the probe beam:

$$C = \frac{3\lambda^2}{4\pi S} \frac{n}{T} \tag{41}$$

T is the intensity transmission of the coupling mirror:

$$T = 2\kappa\tau_C \tag{42}$$

We then derive the relation between the input intensity and the intracavity intensity:

$$\frac{4}{T} \left|\beta^{\mathrm{in}}\right|^2 = \left|\beta_{\mathrm{m}}\right|^2 \left| (1 + i\phi) + 2C \frac{F\left(s_{\mathrm{m}}\right)}{1 + i\delta} \right|^2 \tag{43}$$

or:

$$Y = X \left[\left(1 + CF'\left(X\right)\right)^2 + \left(\phi - \delta CF'\left(X\right)\right)^2 \right] \tag{44}$$

where we have replaced the field parameters by normalized intensities X and Y defined from the intensities averaged over the Gaussian profile:

$$Y = \frac{2}{T} \left|\beta^{\mathrm{in}}\right|^2 \tag{45}$$

$$X = \frac{1}{2} \left|\beta_{\mathrm{m}}\right|^2 = \frac{1}{2} s_{\mathrm{m}} \left(1 + \delta^2\right) \tag{46}$$

$$F'\left(X\right) = \frac{2F\left(s_{\mathrm{m}}\right)}{1 + \delta^2} = \frac{1}{2X} \log\left(1 + \frac{4X}{1 + \delta^2}\right) \tag{47}$$

Equation (44) is identical with results of [17, 18, 19, 20, 21, 23]. It must be compared with the corresponding equation in the plane-wave model:

$$Y = X \left[\left(1 + CF'_{\mathrm{pw}}\left(X\right)\right)^2 + \left(\phi - \delta CF'_{\mathrm{pw}}\left(X\right)\right)^2 \right] \tag{48}$$

where the normalized intensities X and Y are now defined from the position-independent intensities in the plane-wave model:

$$Y = \frac{4}{T} \left|\beta^{\mathrm{in}}\right|^2 \tag{49}$$

$$X = \left|\beta_{\mathrm{e}}\right|^2 = s_{\mathrm{e}} \left(1 + \delta^2\right) \tag{50}$$

$$F'_{\mathrm{pw}}(X) = \frac{2F_{\mathrm{pw}}(s_{\mathrm{e}})}{1+\delta^2} = \frac{2}{1+\delta^2+2X} \tag{51}$$

As long as the saturation parameter remains small ($X \ll 1+\delta^2$), the bistability curves (44) and (48) are identical. Note that this regime corresponds to a low-saturation limit where the mean population of the atomic excited state defined in the plane-wave model remains small:

$$\frac{X}{1+\delta^2+2X} \ll 1 \tag{52}$$

In the following, we will focus the attention on this regime where the predictions of the plane-wave and Gaussian-wave theories coincide for the mean fields. As already emphasized in the introduction, this does not imply that the predictions will also coincide for the fluctuations.

Before coming to the study of fluctuations, we want to give conditions for obtaining a large squeezing. A first condition is that a bistability turning point is approached ([4] and references therein). This condition can be formulated in more quantitative terms by writing the slope of the bistability curve:

$$\frac{dY}{dX} = (1+a)^2 + (\phi - a\delta)^2 - \zeta^2 \tag{53}$$

where:

$$a = \frac{2C\left(1+\delta^2\right)}{\left(1+\delta^2+2X\right)^2} \tag{54}$$

$$\zeta = \frac{4CX\sqrt{1+\delta^2}}{\left(1+\delta^2+2X\right)^2} \tag{55}$$

The coefficient a describes the damping of intracavity field due to atomic absorption, normalized to the damping due to the coupling mirror while the coefficient ζ represents the normalized strength of the Kerr coupling due to atomic nonlinearity. The presence of a turning point ($dY/dX = 0$) requires that the Kerr coupling is large enough so that third term in the right-hand side of equation (53) is larger than the first one:

$$\zeta \geq 1 + a \tag{56}$$

and then that the detuning ϕ of empty cavity is chosen such as to reach one of the two turning points:

$$\phi - a\delta = \pm\sqrt{\zeta^2 - (1+a)^2} \tag{57}$$

The second condition for getting a large squeezing is that bistability is mainly in the dispersive regime, which means that the effect of atomic absorption on the field remains small:

$$a \ll 1 \tag{58}$$

If this condition is not satisfied, atoms efficiently couple vacuum fluctuations coming from the empty modes responsible for spontaneous emission into the intracavity field fluctuations, leading to a degradation of the expected squeezing.

In the low-saturation regime ($X \ll 1+\delta^2$), conditions (58,56) imply a large atom-laser detuning ($\delta^2 \gg 1$) and they may be rewritten:

$$\delta \leq 2C \ll \delta^2 \tag{59}$$

$$\frac{\delta^3}{4C} \leq X \tag{60}$$

In the following, we will consider that these conditions delineate the region where a large squeezing may be obtained. It is in particular worth noting that the intensity parameter X lies in the domain $\delta \ll X \ll \delta^2$.

4 Description of fluctuations in a Gaussian-wave theory

We are now going to study the quantum fluctuations of the field. We will closely follow the reference [16] where quantum fluctuations were derived from a linear input-output theory with a plane-wave model. As in the previous analysis of mean values, the reflection-transmission equations for the field fluctuations on the coupling mirror are the same as in the plane-wave model. In contrast, the effect of the atoms on these fluctuations is modified, due to the non-linearity of the interaction and to the Gaussian profile of laser intensity.

In order to discuss the transformation of field fluctuations, we now introduce a new notation for the positive-frequency and negative-frequency components $E(\mathbf{r},t)$ and $E^{\dagger}(\mathbf{r},t)$ of the field operator (see equation (1)) which are coupled by the nonlinear interaction. They will be considered as the components $\mathcal{E}_{\alpha}$ of a two-fold column matrix $\mathcal{E}$:

$$\mathcal{E}(\mathbf{r},t) = \begin{bmatrix} E(\mathbf{r},t) \\ E^{\dagger}(\mathbf{r},t) \end{bmatrix} \tag{61}$$

We will also denote $\mathcal{E}^{\alpha} = \mathcal{E}_{\alpha}^{\dagger}$ the components of the adjoint line matrix $\mathcal{E}^{\dagger}$.

The field components are written in terms of mode amplitudes and Gaussian mode functions:

$$\mathcal{E}_{\alpha}(\mathbf{r},t) = \sqrt{\frac{\hbar\omega_L}{2\varepsilon_0 c}}\mathcal{A}_{\alpha}(t)\,\mathcal{U}_{\alpha}(\mathbf{r}) \tag{62}$$

The two components $\mathcal{U}_{\alpha}(\mathbf{r})$ coincide with the functions $u(\mathbf{r})$ and $u(\mathbf{r})^*$ defined previously in equation (2-4):

$$\mathcal{U}_\alpha(\mathbf{r}) = \sqrt{\frac{f(\mathbf{r})}{S(x)}}\exp\left(-i\varepsilon_\alpha\varphi(\mathbf{r})\right) \tag{63}$$

$$\varepsilon_\alpha = (-1)^{1+\alpha} \tag{64}$$

The same notation will be used also for the positive-frequency and negative-frequency mode amplitudes:

$$\mathcal{A}(t) = \begin{bmatrix} A(t) \\ A^\dagger(t) \end{bmatrix} \tag{65}$$

as well as for the components of the atomic dipole measured as a dimensionless number:

$$\mathcal{S}(t) = \begin{bmatrix} S(t) \\ S^\dagger(t) \end{bmatrix} \tag{66}$$

($S(t)$ and $S^\dagger(t)$ have the same definition as in [16]; they will be denoted $\mathcal{S}_\alpha(t)$ in the following).

As for the mean values, the effect of an optically thin atomic layer of length dx may be described by a local modification of the field fluctuations:

$$\delta\mathcal{E}_\alpha(\mathbf{r},t) \to \delta\mathcal{E}_\alpha(\mathbf{r},t) + d\left(\delta\mathcal{E}_\alpha(\mathbf{r},t)\right) \tag{67}$$

We have introduced here a generic notation for the fluctuations δO of an operator O:

$$\delta O = O - \langle O\rangle \tag{68}$$

The field modification $d(\delta\mathcal{E}_\alpha)$ may be separated into two parts [16] representing respectively the linear response $d\left(\delta\mathcal{E}_\alpha^{\mathrm{lr}}\right)$ of the atomic dipoles to the intracavity field fluctuations and the spontaneous emission noise $d(\delta\mathcal{E}_\alpha^{\mathrm{se}})$, that is the linear response of atomic dipoles to the vacuum fluctuations in the field modes coupled to the atoms which are different from the intracavity modes [28, 29]:

$$d\left(\delta\mathcal{E}_\alpha(\mathbf{r},t)\right) = d\left(\delta\mathcal{E}_\alpha^{\mathrm{lr}}(\mathbf{r},t)\right) + d\left(\delta\mathcal{E}_\alpha^{\mathrm{se}}(\mathbf{r},t)\right) \tag{69}$$

The spontaneous emission term is uncorrelated with the intracavity field fluctuations. Note that we disregard the effect of the higher-order intracavity modes, in consistency with our assumptions on the treatment of transverse mode structure. We now evaluate these two terms.

The first term may be written in terms of linear susceptibility functions χ_α^β:

$$d\left(\delta\mathcal{E}_\alpha^{\mathrm{lr}}(\mathbf{r},t)\right) = -\frac{3\lambda^2}{4\pi}\rho(\mathbf{r})\,dx\int\sum_\beta \chi_\alpha^\beta(\mathbf{r},\tau)\,\delta\mathcal{E}_\beta(\mathbf{r},t-\tau)\,d\tau \tag{70}$$

The susceptibility function $\chi_\alpha^\beta(\mathbf{r},\tau)$ is deduced from commutators of the atomic dipoles evaluated at different times:

$$\chi_\alpha^\beta(\mathbf{r},\tau) = i\Theta(\tau)\left\langle\left[\mathcal{S}_\alpha(\tau),\mathcal{S}^\beta(0)\right]\right\rangle_{\mathbf{r}} \tag{71}$$

$\Theta(\tau)$ is the Heaviside function. The symbol $\langle ...\rangle_{\mathbf{r}}$ means that the correlation functions are evaluated from the solution of the optical Bloch equations for an atom located at point $\mathbf{r}$. As usually, the optical Bloch equations are solved in the frequency domain rather than in the time domain. We will give below the solution for the Fourier transforms $\chi_\alpha^\beta(\mathbf{r},[\omega])$ of $\chi_\alpha^\beta(\mathbf{r},\tau)$ defined according to the general prescription:

$$f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} f[\omega]$$

The field modification $d\left(\delta\mathcal{E}_\alpha^{\mathrm{lr}}\right)$ will then be written in the frequency domain:

$$d\left(\delta\mathcal{E}_\alpha^{\mathrm{lr}}(\mathbf{r},[\omega])\right) = -\frac{3\lambda^2}{4\pi}\rho(\mathbf{r})\,dx\sum_\beta \chi_\alpha^\beta(\mathbf{r},[\omega])\,\delta\mathcal{E}_\beta(\mathbf{r},[\omega]) \tag{72}$$

Clearly, the susceptibility function will depend upon the position of the atom because of the Gaussian mode structure of the field, which will make a difference with the plane-wave model of reference [16].

Before considering this problem of position-dependence of the susceptibility function, we come to characterize the second fluctuating term $d\left(\delta\mathcal{E}_\alpha^{\mathrm{se}}\right)$ appearing in equation (69), which represents the spontaneous emission noise. Assuming a dilute atomic medium such that the fluctuations of the different dipoles are independent of each other, we obtain field fluctuations by summing up the contributions of all dipoles:

$$\left\langle d\left(\delta\mathcal{E}_\alpha^{\mathrm{se}}(\mathbf{r},[\omega])\right) d\left(\delta\mathcal{E}^{\beta\ \mathrm{se}}(\mathbf{r}',[\omega'])\right)\right\rangle = 2\pi\delta(\omega+\omega')\,\delta(\mathbf{r}-\mathbf{r}')\,\frac{3\lambda^2}{4\pi}\rho(\mathbf{r})\,dx\,\sigma_\alpha^\beta(\mathbf{r},[\omega]) \tag{73}$$

The correlation functions σ_α^β describe the fluctuations of the dipole components evaluated for one atom at point $\mathbf{r}$:

$$\sigma_\alpha^\beta(\mathbf{r},\tau) = \left\langle\delta\mathcal{S}_\alpha(\tau)\,\delta\mathcal{S}^\beta(0)\right\rangle_{\mathbf{r}} \tag{74}$$

The analytical expressions of the Fourier transforms $\sigma_\alpha^\beta(\mathbf{r},[\omega])$, as well as those of $\chi_\alpha^\beta(\mathbf{r},[\omega])$, can be obtained in terms of a single atomic function G_α^β which will be given in the next section.

As already carried out for the mean values, we now solve the problem of the transverse dependence of correlation functions by supposing that the fluctuations radiated by the atoms into the higher-order modes have a negligible effect on intracavity field fluctuations, and furthermore are not matched to the

detection setup. We will therefore obtain the modification $d\left(\delta\mathcal{A}_\alpha\right)$ of the amplitude fluctuations due to the atomic medium by projecting the field fluctuations $d\left(\delta\mathcal{E}_\alpha\right)$ onto the Gaussian mode functions:

$$d\left(\delta\mathcal{A}_\alpha\left(t\right)\right)=\int d\left(\delta\mathcal{E}_\alpha\left(\mathbf{r},t\right)\right)\mathcal{U}_\alpha^*\left(\mathbf{r}\right)dy\ dz$$

Considering first the linear response term, we project the effect of a thin atomic layer of length dx (cf. equation (70)) and sum up over the whole atomic medium. We thus write $d\left(\delta\mathcal{A}_\alpha^{\mathrm{lr}}\right)$ in the frequency domain in terms of a new susceptibility function $\overline{\chi}_\alpha^\beta\left[\omega\right]$:

$$d\left(\delta\mathcal{A}_\alpha^{\mathrm{lr}}\left[\omega\right]\right)=-\frac{3\lambda^2}{4\pi S}n\sum_\beta\overline{\chi}_\alpha^\beta\left[\omega\right]\delta\mathcal{A}_\beta\left[\omega\right] \tag{75}$$

which is defined as the local susceptibility function averaged over the beam profile:

$$\overline{\chi}_\alpha^\beta\left[\omega\right]=\frac{1}{n}\int\chi_\alpha^\beta\left(\mathbf{r},\left[\omega\right]\right)\rho\left(\mathbf{r}\right)f\left(\mathbf{r}\right)e^{i(\varepsilon_\alpha-\varepsilon_\beta)\varphi(\mathbf{r})}d\mathbf{r} \tag{76}$$

For the spontaneous emission noise, we once more disregard the fluctuations radiated by the atoms into the higher-order modes, then project the field fluctuations $d\delta\mathcal{E}_\alpha^{\mathrm{se}}$ onto the Gaussian mode, and finally sum up the independent contributions of all atoms. We thus obtain the contribution of spontaneous emission noise in terms of a correlation function $\overline{\sigma}_\alpha^\beta\left[\omega\right]$ averaged over the transverse profile of the beam:

$$\left\langle d\left(\delta\mathcal{A}_\alpha^{\mathrm{se}}\left[\omega\right]\right)d\left(\delta\mathcal{A}^{\beta\ \mathrm{se}}\left[\omega'\right]\right)\right\rangle=2\pi\delta\left(\omega+\omega'\right)\frac{3\lambda^2}{4\pi S}n\overline{\sigma}_\alpha^\beta\left[\omega\right] \tag{77}$$

$$\overline{\sigma}_\alpha^\beta\left[\omega\right]=\frac{1}{n}\int\sigma_\alpha^\beta\left(\mathbf{r},\left[\omega\right]\right)\rho\left(\mathbf{r}\right)f\left(\mathbf{r}\right)e^{i(\varepsilon_\alpha-\varepsilon_\beta)\varphi(\mathbf{r})}d\mathbf{r} \tag{78}$$

Now, the computation of quantum fluctuations in the output field proceeds along the same lines as in the plane-wave model [16], provided that the following modifications are accounted for. First, field fluctuations are understood as refering to the Gaussian mode amplitudes, for the intracavity field $\delta\mathcal{A}_\alpha$ as well as for input and output fields $\delta\mathcal{A}_\alpha^{\mathrm{in}}$ and $\delta\mathcal{A}_\alpha^{\mathrm{out}}$. Then, the plane-wave susceptibility function χ_α^β and noise spectrum σ_α^β are replaced by the averaged expressions $\overline{\chi}_\alpha^\beta$ and $\overline{\sigma}_\alpha^\beta$ given respectively by equations (76) and (78). This leads to a simple and natural generalisation of the results of the plane-wave model. When transverse beam structure is accounted for, the position-dependent expressions for atomic susceptibility function and atomic noise spectrum have to be averaged over the beam intensity profile. The same property was in fact used in published calculations accounting for Gaussian profile of the laser beam [21, 23]: in these papers where quantum stochastic methods were used, the squeezing

spectra were obtained by averaging the drift and diffusion matrices which describe in these methods the effect of atoms. In the present paper, the fact that the contributions of elementary volumes have to be added independently has not been postulated. We have shown that this averaging property follows from a simple treatment of Gaussian profile where fields radiated by the atoms are projected onto the fundamental Gaussian mode.

5 Evaluation of the averaged atomic spectra

Before coming to the estimation of the squeezing spectra, we now give the expressions of the atomic spectra and deduce the expressions averaged over the intensity profile. We then compare the atomic spectra thus obtained with those derived in the plane-wave model.

The local susceptibility function and noise spectrum $\chi_\alpha^\beta(\mathbf{r},[\omega])$ and $\sigma_\alpha^\beta(\mathbf{r},[\omega])$ are obtained by solving optical Bloch equations for an atom located at point $\mathbf{r}$. They have the same form as in the plane-wave model [15, 16], with a position-dependent field parameter $\beta(\mathbf{r})$ however, and may be written in terms of the phase $\varphi(\mathbf{r})$ of the Gaussian mode and of a function G_α^β which only depends on the local intensity parameter:

$$\chi_\alpha^\beta(\mathbf{r},[\omega]) = \frac{i}{2\gamma}\left(G_\alpha^\beta(\mathbf{r},[\omega]) - G_{3-\alpha}^{3-\beta}(\mathbf{r},[-\omega])^*\right)e^{-i(\varepsilon_\alpha-\varepsilon_\beta)\varphi(\mathbf{r})} \quad (79)$$

$$\sigma_\alpha^\beta(\mathbf{r},[\omega]) = \frac{1}{2\gamma}\left(G_\alpha^\beta(\mathbf{r},[\omega]) + G_\beta^\alpha(\mathbf{r},[\omega])^*\right)e^{-i(\varepsilon_\alpha-\varepsilon_\beta)\varphi(\mathbf{r})} \quad (80)$$

$$G_\alpha^\beta(\mathbf{r},[\omega]) = \frac{p_\alpha^\beta\left(|\beta(\mathbf{r})|^2,[\omega]\right)}{q\left(|\beta(\mathbf{r})|^2,[\omega]\right)} \quad (81)$$

where:

$$\begin{aligned} p_1^1\left(|\beta|^2,[\omega]\right) &= \left(1+\delta^2+2|\beta|^2\right)\left(1+\delta^2\right)(1-i\delta-i\overline{\omega})(2-i\overline{\omega}) \\ &\quad +2|\beta|^2\left(1+\delta^2+2|\beta|^2\right)i\overline{\omega}(1+i\delta) \\ &\quad +2|\beta|^4(2-i\overline{\omega})^2+4|\beta|^6 \end{aligned} \quad (82)$$

$$\begin{aligned} p_2^1\left(|\beta|^2,[\omega]\right) &= 2|\beta|^2\left(1+\delta^2+2|\beta|^2\right)(2-i\overline{\omega})(1+i\delta) \\ &\quad +|\beta|^2(1+i\delta)^2(1+i\delta-i\overline{\omega})(2-i\overline{\omega}) \\ &\quad +2|\beta|^4 i\overline{\omega}(1+i\delta)+4|\beta|^6 \end{aligned} \quad (83)$$

$$p_1^2\left(|\beta|^2,[\omega]\right) = |\beta|^2(1-i\delta)^2(1-i\delta-i\overline{\omega})(2-i\overline{\omega})$$

$$+2\left|\beta\right|^{4} i\overline{\omega}\left(1-i\delta\right)+4\left|\beta\right|^{6} \tag{84}$$

$$p_{2}^{2}\left(\left|\beta\right|^{2},\left[\omega\right]\right) = 2\left|\beta\right|^{4}\left(2-i\overline{\omega}\right)^{2}+4\left|\beta\right|^{6} \tag{85}$$

$$\begin{aligned} q\left(\left|\beta\right|^{2},\left[\omega\right]\right) &= \left(1+\delta^{2}+2\left|\beta\right|^{2}\right)^{2}\left(2-i\overline{\omega}\right)\left(\left(1-i\overline{\omega}\right)^{2}+\delta^{2}\right) \\ &+\left(1+\delta^{2}+2\left|\beta\right|^{2}\right)^{2}4\left|\beta\right|^{2}\left(1-i\overline{\omega}\right) \end{aligned} \tag{86}$$

δ and β are the dimensionless parameters already defined for measuring laser-atom detuning and Rabi frequency respectively; $\overline{\omega}$ is the noise frequency also normalized to the decay rate of the dipole:

$$\overline{\omega}=\frac{\omega}{\gamma} \tag{87}$$

In order to perform the average over the beam profile, we first note that the phases do not enter the averaged expressions (compare equations (76,78) with equations (79,80)). We then proceed by analogy with the computation of the factors $n^{(1+k)}$ which appeared in the expression (27) of the effective number of atoms, by developing the functions G_{α}^{β} in terms of the intensity parameter:

$$G_{\alpha}^{\beta}\left(\mathbf{r},\left[\omega\right]\right)=\sum_{k=0}^{\infty}G_{\alpha}^{\beta\ (k)}\left[\omega\right]\left|\beta\left(\mathbf{r}\right)\right|^{2k} \tag{88}$$

Writing the position-dependent local intensity as in the previous discussion of the bistability curve:

$$\left|\beta\left(\mathbf{r}\right)\right|^{2}=\left|\beta_{\mathrm{m}}\right|^{2}f\left(\mathbf{r}\right)=2Xf\left(\mathbf{r}\right) \tag{89}$$

we obtain:

$$\overline{\chi}_{\alpha}^{\beta}\left[\omega\right] = \sum_{k=0}^{\infty}\frac{i\xi_{k}X^{k}}{2\gamma}\left(G_{\alpha}^{\beta\ (k)}\left[\omega\right]-G_{3-\alpha}^{3-\beta\ (k)}\left[-\omega\right]^{*}\right) \tag{90}$$

$$\overline{\sigma}_{\alpha}^{\beta}\left[\omega\right] = \sum_{k=0}^{\infty}\frac{\xi_{k}X^{k}}{2\gamma}\left(G_{\alpha}^{\beta}\left[\omega\right]+G_{\beta}^{\alpha\ (k)}\left[\omega\right]^{*}\right) \tag{91}$$

where:

$$\xi_{k}=\frac{2^{k}}{1+k} \tag{92}$$

The plane-wave expressions $\chi_{\alpha}^{\beta}\left[\omega\right]$ and $\sigma_{\alpha}^{\beta}\left[\omega\right]$ written in equations (79,80) may be identified with equations (79,80) with ξ_{k} replaced by unity (global phase factors are unessential and may be disregarded). The definitions of X, standing for the position-independent intensity parameter $\left|\beta\right|^{2}$ in the plane-wave model

and for its spatial average over the beam profile in the Gaussian-wave theory, have been precisely chosen so that the coefficients ξ_0 and ξ_1 are equal to unity. This implies that the differences between the plane-wave and Gaussian-beam predictions for squeezing only come from the fact that the higher-order coefficients ξ_k (with $k = 2, 3 \ldots$) differ from unity. The discussion of these differences may therefore be restricted to the analysis of the contributions to atomic spectra proportional to X^2, $X^3 \ldots$: the plane-wave predictions cannot be considered as reliable as soon as these contributions become appreciable.

It could be expected that the contributions proportional to X^2, $X^3 \ldots$ to the atomic spectra become appreciable, as for the mean values, only when the excited state becomes significantly populated, that is when the intensity parameter X approaches δ^2. The plane-wave predictions would thus be reliable in the whole range $\delta < X < \delta^2$ where a large squeezing is expected (see the discussion at the end of section 3). We now show that this is not the case.

We first consider the limiting case, that we will call the Kerr domain in the following, where only the terms proportional to X^0 and X^1 are appreciable. In this case, the coincidence between the atomic spectra obtained in the two models is guaranteed. The linear terms (proportional to X^0) are given by:

$$G_1^{1\ (0)} = \frac{1}{1 + i\delta - i\overline{\omega}} \tag{93}$$

$$G_2^{1\ (0)} = G_1^{2\ (0)} = G_2^{2\ (0)} = 0 \tag{94}$$

and the Kerr terms (proportional to X^1) by:

$$G_1^{1\ (1)} = \frac{-2\,(2 + 2i\delta - i\overline{\omega})}{(1+\delta^2)\,(1 + i\delta - i\overline{\omega})^2} \tag{95}$$

$$G_2^{1\ (1)} = \frac{2}{(1 - i\delta)\left((1 - i\overline{\omega})^2 + \delta^2\right)} + \frac{1}{(1 - i\delta)^2\,(1 + i\delta - i\overline{\omega})} \tag{96}$$

$$G_1^{2\ (1)} = \frac{1}{(1 + i\delta)^2\,(1 + i\delta - i\overline{\omega})} \tag{97}$$

$$G_2^{2\ (1)} = 0 \tag{98}$$

These terms have a simple physical interpretation in terms of a diagrammatic perturbative expansion of the field-matter interaction [30, 31]. The linear terms are described by two-photon elastic scattering amplitudes which are resonant at frequencies close to the atomic eigenfrequency ($\overline{\omega} \simeq \delta$) while the Kerr terms are described by four-photon inelastic scattering amplitudes which are resonant at frequencies either close to the atomic eigenfrequency ($\overline{\omega} \simeq \delta$) or symmetric to the atomic eigenfrequency with respect to the laser frequency ($\overline{\omega} \simeq -\delta$). The interpretation of squeezing in terms of these photon scattering amplitudes was already performed for the case of atoms coupled to field in free space [26, 31].

The previous equations give a more complete description of the effect of these amplitudes which may be used also for interpreting squeezing by atoms coupled to field in a cavity. An important consequence of this discussion is that the squeezing processing is resonant on the sidebands of fluorescence triplet ($\overline{\omega} \simeq \pm\delta$) and not on the central component ($\overline{\omega} \simeq 0$). In contrast, squeezing via bistability is usually studied for frequencies close to the laser frequency. This is particularly clear in the good-cavity limit where the entire cavity bandwidth is contained in the central component of the emission triplet. In this case, the previous equations have to be evaluated at zero frequency ($\overline{\omega} = 0$) where they correspond to non-resonant processes.

We come now to the analysis of the higher-order contributions to atomic spectra proportional to X^2, $X^3 \ldots$, and focus the discussion on the same regime where squeezing is studied for frequencies close to the laser frequency. In this regime, it is easily checked on the expressions of the atomic susceptibility functions χ_α^β that the higher-order contributions remain negligible as long as $X \ll \delta^2$. This is related to the general property of linear response theory which states that static susceptibilities evaluated at $\overline{\omega} = 0$ are directly related to differentiated forms of the relations between mean fields, and to the already known fact that the plane-wave model provides reliable expressions for mean fields when $X \ll \delta^2$. However, the higher-order contributions to the noise spectra σ_α^β may become appreciable even when $X \ll \delta^2$. This can be shown by considering the atomic spectra at zero frequency (obtained by setting $\overline{\omega} = 0$ in the unapproximated forms (81)):

$$G_1^1[0] = \frac{\left(1+\delta^2\right)(1-i\delta)}{\left(1+\delta^2+2|\beta|^2\right)^2} + \frac{4|\beta|^4+2|\beta|^6}{\left(1+\delta^2+2|\beta|^2\right)^3} \tag{99}$$

$$G_2^1[0] = \frac{2|\beta|^2(1+i\delta)}{\left(1+\delta^2+2|\beta|^2\right)^2} + \frac{|\beta|^2(1+i\delta)^3+2|\beta|^6}{\left(1+\delta^2+2|\beta|^2\right)^3} \tag{100}$$

$$G_1^2[0] = \frac{|\beta|^2(1-i\delta)^3+2|\beta|^6}{\left(1+\delta^2+2|\beta|^2\right)^3} \tag{101}$$

$$G_2^2[0] = \frac{4|\beta|^4+2|\beta|^6}{\left(1+\delta^2+2|\beta|^2\right)^3} \tag{102}$$

The evaluation of the susceptibility spectra relies only upon the first terms appearing in $G_1^1[0]$ and $G_2^1[0]$, so that the higher-order contributions to these terms are negligible when $X \ll \delta^2$. In contrast, the other terms, which enter in the evaluation of the noise spectra, may be appreciable even for $X \ll \delta^2$. Restricting our interest to this low-saturation limit, we may identify the main contribution to the squeezing process as the parametric terms proportional to

X/δ^3 which appear in $G_2^1[0]$. Among the dominant noise contributions, there are the terms proportional to $1/\delta^2$ in $G_1^1[0]$ and X/δ^4 in $G_2^1[0]$ or $G_1^2[0]$, but also the higher-order terms proportional to X^2/δ^6 or X^3/δ^6 which appear in the four functions $G_\beta^\alpha[0]$. It is thus clear that these terms may not be disregarded for $X \ll \delta^2$. In particular, the noise term proportional to X^3/δ^6 reaches the magnitude of the parametric Kerr coefficient X/δ^3 for $X \approx \delta^{3/2}$.

This noise term is not accounted for in a Kerr model and is found to be responsible for excess noise and degradation of the expected squeezing [8]. From the qualitative evaluations of previous paragraph, one deduces that the noise correlation functions have to be suspected in the domain $\delta < X < \delta^2$, where a large squeezing is expected from plane-wave computations (see conditions (59,60)). This does not imply that plane-wave predictions of squeezing are wrong in the whole domain, but this forces us to perform the Gaussian-wave computations in order to get reliable expectations. In the next section, we give a quantitative estimate of squeezing degradation due to this effect and delineate the domain of validity of plane-wave computations.

The conclusion of this discussion is that, although the plane-wave model provides reliable estimations for the bistability curve and for the susceptibility functions in the whole low-saturation domain $X \ll \delta^2$, this is not the case for the noise correlation functions. A qualitative interpretation of this result may be found by coming back to photon scattering amplitudes. As already discussed, the two-photon and four-photon amplitudes are resonant on the sidebands of the fluorescence triplet. In contrast, there exists higher-order scattering amplitudes which give rise to inelastic fluorescence on the central component of the triplet [30]. Although these amplitudes are proportional to higher-order powers of X, they are favored with respect to the lowest-order amplitudes, due to their resonant enhancement around the central frequency. This explains why they can have an influence in the low-saturation domain.

6 Squeezing spectra

We come now to the computation of quantum fluctuations in the output field. As already discussed, this computation proceeds along the same lines in the Gaussian-wave computations as in the plane-wave model, provided that the matrices χ_α^β and σ_α^β are replaced by the averaged expressions $\overline{\chi}_\alpha^\beta$ and $\overline{\sigma}_\alpha^\beta$.

The covariance functions for the input or output fields are defined in terms of noise matrices:

$$\left\langle \delta\mathcal{A}_\alpha^{\text{in}}[\omega]\, \delta\mathcal{A}^{\beta\ \text{in}}[\omega'] \right\rangle = 2\pi\delta(\omega+\omega')\, \mathcal{V}_\alpha^{\beta\ \text{in}}[\omega] \tag{103}$$

$$\left\langle \delta\mathcal{A}_\alpha^{\text{out}}[\omega]\, \delta\mathcal{A}^{\beta\ \text{out}}[\omega'] \right\rangle = 2\pi\delta(\omega+\omega')\, \mathcal{V}_\alpha^{\beta\ \text{out}}[\omega] \tag{104}$$

$\mathcal{V}_\alpha^{\beta\ \text{in}}$ and $\mathcal{V}_\alpha^{\beta\ \text{out}}$ are the elements of square matrices $\mathcal{V}^{\text{in}}$ and $\mathcal{V}^{\text{out}}$ which are connected through noise processing relations (equations (47-51) of reference [16]):

$$\mathcal{V}^{\text{out}}[\omega] = (\mu[\omega] - 1).\mathcal{V}^{\text{in}}[\omega].(\mu[\omega] - 1)^{\dagger} + \mu[\omega].\mathcal{V}^{\text{at}}[\omega].\mu[\omega]^{\dagger} \tag{105}$$

$(\mu[\omega] - 1)$ and $\mu[\omega]$ are transfer matrices which characterize noise processing by the cavity containing atoms and which depends on the susceptibility matrix $\overline{\chi}$:

$$\mu[\omega] = \frac{2}{1 + i\phi - i\frac{\omega}{\kappa} - 2\gamma C\varepsilon.\overline{\chi}[\omega]} \tag{106}$$

$\mathcal{V}^{\text{at}}$ describes the noise added by the atomic medium, and which depends on the atomic noise matrix $\overline{\sigma}$:

$$\mathcal{V}^{\text{at}}[\omega] = \gamma C\varepsilon.\overline{\sigma}[\omega].\varepsilon \tag{107}$$

In these expressions, ε is a diagonal matrix containing the coefficients ε_α:

$$\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{108}$$

It is worth stressing at this stage that, while the atomic spectra $\overline{\chi}$ and $\overline{\sigma}$ are obtained by averaging the plane-wave spectra over the intensity profile, this is not the case for the noise spectra of the output fields.

We may now give a quantitative estimation of the influence of the Gaussian profile upon the noise spectra for the output fields. The transfer matrices are not modified in the low-saturation limit, since they depend only upon the susceptibility matrix $\overline{\chi}$. It follows that the first term in the right-hand side of equation (105) is not influenced by the Gaussian profile. In contrast, the second term, which is proportional to the atomic noise matrix $\overline{\sigma}$, has a larger degrading effect on squeezing in the Gaussian-wave theory than in the plane-wave model. Precisely, the squeezing degradation due to higher-order noise terms is greater in Gaussian-wave theory than in the plane-wave model; for terms proportional to X^2 (respectively X^3), this degradation is multiplied by $\xi_2 = 4/3$ (respectively by $\xi_3 = 2$). In the Kerr domain, those higher-order noise terms have a negligible influence on squeezing, so that the predictions of plane-wave model coincide with those of Gaussian-wave theory.

A more detailed discussion may be given by studying the example of the optimum squeezing expected at zero frequency ($\omega = 0$) on a bistability turning point (defined by equations (56,57)). The optimum squeezing (variance of the quadrature component with minimal fluctuations, evaluated at zero frequency on a bistability turning point) reaches in the Kerr domain the value:

$$S_{\text{opt}}^{\text{Kerr}} = \frac{a}{1 + a} \tag{109}$$

This value is small, of the order of $2C/\delta^2$, when condition (59) is obeyed, that is when the effect of atomic absorption on the field remains small. Note that

a small value of $S_{\rm opt}$ means an efficient squeezing of input field fluctuations by the bistable cavity. Assuming that the intensity parameter X lies in the domain $\delta \ll X \ll \delta^2$, a close inspection of the equations describing noise processing by the bistable cavity shows that the optimum squeezing $S_{\rm opt}$ accounting for higher-order noise terms in the plane-wave model may be written:

$$S_{\rm opt} \simeq \frac{S_{\rm opt}^{\rm Kerr} + 2f\frac{X^2}{\delta^3}}{1 + 2f\frac{X^2}{\delta^3}} \tag{110}$$

The numerical factor f has the following value, of the order of unity (ϕ is defined by (57)):

$$f = \frac{\zeta - (\phi - a\delta)}{1 + a} \tag{111}$$

The degradation due to higher-order noise terms is thus found to become appreciable when $2X^2/\delta^3 \simeq 2C/\delta^2$, that is when:

$$X \simeq \sqrt{C\delta} \tag{112}$$

The degradation of squeezing appearing in equation (110) is enlarged when passing from the plane-wave to the Gaussian-wave theory. It turns out that, in the limiting case where degradation is small, it may be attributed mainly to higher-order atomic noise terms proportional to X^3, so that squeezing degradation is roughly multiplied by $\xi_3 = 2$ due to Gaussian transverse profile. In other words, the plane-wave model systematically underestimates the degrading effect of higher-order contributions to atomic noise and does not provide reliable estimations for the squeezing when such contributions cannot be neglected.

In contrast, we may consider now the condition which delineates the Kerr region:

$$X \ll \sqrt{C\delta} \tag{113}$$

When this condition is satisfied, the optimum squeezing is not degraded by higher-order noise terms, so that the plane-wave computation provides a reliable estimation for squeezing. It is worth recalling here that this condition has been derived in the domain $\delta \ll X \ll \delta^2$ where a large squeezing is expected. In the domain of very low intensity $X < \delta$, different conclusions would be drawn for reliability of the plane-wave model, but a poor squeezing would be predicted.

The previous discussion was restricted to the particular case of zero frequency where analytical expressions may be handled more easily. Figure 1 shows spectra for the optimum squeezing evaluated as a function of noise frequency ω. The three spectra correspond to the same value for the atomic absorption coefficient a and to the same working point on the bistability curve; we have chosen the bistability threshold such that there is only one turning point, which corresponds to $\zeta = 1 + a$ and $\phi = a\delta$ ($f = 1$ in this particular case).

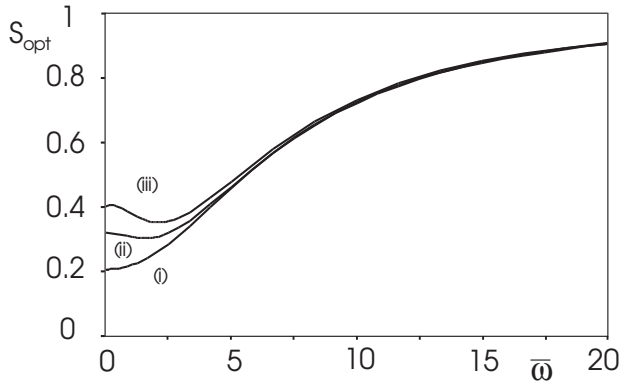

Figure 1: Spectra for the optimum squeezing as a function of normalized noise frequency $\overline{\omega} = \omega/\gamma$, with a decay constant for the intracavity field $\kappa = 10\gamma$; the three spectra correspond to the same atomic absorption coefficient ($a = 0.25$) and to the same turning point at bistability threshold ($\zeta = 1 + a$; $\phi = a\delta$); the spectrum (i) is computed for a large detuning lying in the Kerr domain ($\delta = 1000$; the other parameters deduced from the values of δ, a and ζ are $X = 2500$, $C = 126253$, $\phi = 250$); the two other spectra are computed for a lower detuning lying outside the Kerr domain ($\delta = 100$; $X = 250$, $C = 1378$, $\phi = 25$) and result from plane-wave computations (ii) and Gaussian-wave computations (iii).

The first spectrum (trace (i)) is computed for a detuning sufficiently large so that the Kerr limit is reached. In this case, the spectra computed from plane-wave model and Gaussian-wave theory cannot be distinguished. The two other spectra are computed for a lower detuning, such that higher-order noise terms degrade optimum squeezing. The results of plane-wave computations (trace (ii)) and Gaussian-wave computations (trace (iii)) now differ, revealing in particular a larger degradation when transverse structure is accounted for. In addition, these spectra clearly show, in consistency with the discussion at the end of section 5, that the excess noise responsible for squeezing degradation is associated with the central peak of the atomic fluorescence spectrum, whose width equals 2γ, where γ is the decay constant of the atomic dipole, in the limiting case of low saturation [32]. Note that the spectra of Figure 1 have been computed in the bad-cavity limit where the decay constant κ of the intracavity field is larger than γ. In the opposite good-cavity limit, the resonant behaviour of the higher-order excess noise would not be apparent on the squeezing spectrum.

7 Conclusion

We have used the linear input-output formalism generalized to incorporate the treatment of atomic fluctuations to discuss the effect of Gausian transverse structure on squeezing with two-level atoms. We have shown that this effect is described quite naturally as a spatial averaging of the susceptibility and noise functions which appear in the linear input-output formalism. This property has not been postulated a priori, but derived as a consequence of the simple assumptions made in the treatment of transverse profile: higher-order transverse modes have been supposed either far from resonance, or of such a high order that they are not efficiently coupled to the Gaussian mode.

The mean fields and the bistability curves deduced from plane-wave and Gaussian-wave calculations coincide in the low-saturation limit where the population of the atomic excited state remains small ($X \ll \delta^2$). We have shown that this is also the case for the atomic susceptibility functions evaluated around zero frequency. This is related to the general property of linear response theory that the static susceptibilities are differentiated forms of the relations between mean fields. In contrast, the limit of low saturation is not sufficient for ensuring the coincidence of plane-wave and Gaussian-wave predictions for atomic noise functions. This is due to the existence of higher-order noise terms, arising from the central peak of the fluorescence spectrum. These terms are able to degrade the optimum squeezing when X reaches the value $\sqrt{C\delta}$. We have shown that these terms have a larger degrading effect in the Gaussian-wave calculations as in the plane-wave ones. This implies that plane-wave predictions of a large squeezing are reliable only in the Kerr domain, that is for $X \ll \sqrt{C\delta}$. This criterium for reliability of plane-wave model appears quite different from those published in previous references [21, 22, 23].

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References

[1] Lugiato L.A., *in Progress in Optics* **XXI** 71 (1984)

[2] Lugiato L.A. and Strini G., *Optics Communications* **41** 67 (1982); *Optics Communications* **41** 374 (1982)

[3] Collett M.J. and Walls D.F., *Physical Review* **A32** 2887 (1985)

[4] Reynaud S., Fabre C., Giacobino E. and Heidmann A., *Physical Review* **A40** 1440 (1989)

[5] Raizen M.G., Orozco L.A., Min Xiao, Boyd T.L. and Kimble H.J., *Physical Review Letters* **59** 198 (1987)

[6] Hope D.M., Bachor H.A., Manson P.J., McClelland D.E., Fisk P.T.H., *Physical Review* **A46** R1181 (1992)

[7] Lambrecht A., Giacobino E., Courty J.-M. and Reynaud S., *Applied Physics* **B60** 129 (1995)

[8] Reid M.A. and Walls D.F., *Physical Review* **A32** 396 (1985)

[9] Reid M.A. and Walls D.F., *Physical Review* **A33** 4465 (1986)

[10] Reid M.A. and Walls D.F., *Physical Review* **A34** 4929 (1986)

[11] Carmichael H.J., *Physical Review* **A33** 3262 (1986)

[12] Lugiato L.A., *Physical Review* **A33** 4079 (1986)

[13] Orozco L.A., Raizen M.G., Min Xiao, Brecha R.J. and Kimble H.J., *Journal of the Optical Society of America* **B4** 1490 (1987)

[14] Castelli F., Lugiato L.A. and Vadacchino M., *Nuovo Cimento* **B10** 183 (1988)

[15] Reid M.A., *Physical Review* **A37** 4792 (1988)

[16] Hilico L., Fabre C., Giacobino E. and Reynaud S., *Physical Review* **A46** 4397 (1992)

[17] Drummond P.D., *IEEE Journal of Quantum Electronics* **QE17** 301 (1981)

[18] Ballagh R.J., Cooper J., Hamilton M.W., Sandle W.J. and Warrington D.M., *Optics Communications* **37** 143 (1981)

[19] Lugiato L.A. and Milani M., *Zeitschrift fur Physik* **B50** 171 (1983)

[20] Rosenberger A.T., Orozco L.A. and Kimble H.J., *Physical Review* **A28** 2569 (1983)

[21] Xiao M., Kimble H.J. and Carmichael H.J., *Physical Review* **A35** 3832 (1987)

[22] Xiao M., Kimble H.J. and Carmichael H.J., *Journal of the Optical Society of America* **B4** 1546 (1987)

[23] Hope D.M., McClelland D.E. and Savage C.M., *Physical Review* **A41** 5074 (1990)

[24] Collett J. and Gardiner C.W., *Physical Review* **A30** 1386 (1984)

[25] Reynaud S. and Heidmann A., *Optics Communications* **71** 209 (1989)

[26] Heidmann A. and Reynaud S., *Journal de Physique* **46** 1937 (1985)

[27] Lambrecht A., Giacobino E. and Reynaud S., *Atomic number fluctuations in a cold atom cloud* preprint (1995)

[28] Courty J.-M., Grangier P., Hilico L. and Reynaud S., *Optics Communications* **83** 251 (1991)

[29] Courty J.-M. and Reynaud S., *Physical Review* **A46** 2766 (1992)

[30] Dalibard J. and Reynaud S., *Journal de Physique* **44** 1337 (1983)

[31] Heidmann A. and Reynaud S., *Journal of Modern Optics* **34** 923 (1987)

[32] Cohen-Tannoudji C. and Reynaud S., in *Multiphoton Processes*, eds J.H.Eberly and P.Lambropoulos (Wiley, 1978) p.103